Quiz Solutions

1. Let X be a topological space, and Y be a compact topological space.

   (a) Show that the projection map $\pi_1 : X \times Y \to X$ is a closed map.

   (b) Let Y be Hausdorff space, and $f : X \to Y$ be a map. Show that f is continuous if, and only if, the *graph of f*

   $$G_f = \{(x, y) \,|\, (x, y) \in X \times Y \text{ and } y = f(x)\}$$

   is a closed subset of $X \times Y$.

   Solution. (a) Let $A \subset X \times Y$ be a closed subset. We show that $X \setminus \pi_1(A) = \pi_1(A)^c$ is a open subset of X. Now for any $x_0 \in \pi_1(A)^c$, we have that $(x_0, y) \in A^c (= X \times Y \setminus A)$, for any $y \in Y$. Since A^c is open, for each $y \in Y$, there exists a basic open set $U_y \times V_y \subset X \times Y$ such that

   $$(x_0, y) \in U_y \times V_y \subset A^c.$$

   As the collection of open sets $\{V_y \,|\, y \in Y\}$ forms an open cover for Y, the compactness of Y would imply that there exists $y_1, \ldots y_k \in Y$ such that

   $$Y = \bigcup_{i=1}^{k} V_{y_i}.$$

   Now let $U = \cap_{i=1}^{k} U_{y_i}$; then clearly $x_0 \in U$. Moreover, as each $U_{y_i} \times X_{y_i} \subset A^c$, we have that $U \subset \pi_1(A)^c$, and the result follows.

   (b) ($\implies$) Suppose that $f : X \to Y$ is a continuous map. We show that

   $$G_f^c = (X \times Y) \setminus G_f = \{(x, y) \,|\, y \neq f(x)\} = X \times (Y \setminus f(X))$$

   is an open subspace of $X \times Y$. For any $(x, y) \in G_f^c$, as $y \neq f(x)$ and Y is Hausdorff, there exists disjoint open sets $U \ni f(x)$ and $V \ni y$ in Y, so we get $(x, y) \in f^{-1}(U) \times V$. Since f is continuous, $f^{-1}(U) \times V$ is a basic open set in $X \times Y$. It remains to show that $f^{-1}(U) \times V \subset G_f^c$. Suppose we assume on the contrary that $(x', y') \in (f^{-1}(U) \times V) \cap G_f$. Then we have that $f(x') \in U$, $y' \in V$, and $y' = f(x')$. This would imply that $f(x') \in U \cap V$, which contradicts the fact that U and V are disjoint.

($\Longleftarrow$) Conversely, let us assume that G_f is closed in $X \times Y$. For $x \in X$, take an open set $V \ni f(x)$ in Y. Then since $V^c = Y \setminus V$ is closed in Y, we have that $X \times V^c$ is a closed subset of $X \times Y$. Our hypothesis would then imply that

$$G_f \cap (X \times V^c)$$

is a closed subset of $X \times Y$. By Part (a), we have that

$$\pi_1(G_f \cap (X \times V^c))$$

is a closed subset of X. But note that

$$\begin{aligned} \pi_1(G_f \cap (X \times V^c)) &= \{x \mid f(x) \in V^c\} \\ &= f^{-1}(V^c) \\ &= (f^{-1}(V))^c, \end{aligned}$$

which would imply that

$$(\pi_1(G_f \cap (X \times V^c)))^c = f^{-1}(V).$$

Hence, we have that

$$f((\pi_1(G_f \cap (X \times V^c)))^c) = f(f^{-1}(V)) \subset V.$$

Taking

$$U = (\pi_1(G_f \cap (X \times V^c)))^c,$$

we see that $U \ni x$ and $f(U) \subset V$, or in other words, for every neighborhood $V \ni f(x)$, there exists a neighborhood $U \ni x$ such that $f(U) \subset U$. Therefore, f is continuous.

2. (a) State the Urysohn metrization theorem.
   (b) Let X be compact Hausdorff space. Then show that X is metrizable if, and only if, X is second countable.

Solution. (a) The Urysohn metrization theorem states that: A regular second countable topological space is metrizable.

(b) Let X be a compact Hausdorff space. Then from result discussed in class, we know that X has to be normal, and hence regular.

($\Longleftarrow$) Suppose that X is second countable. Then the Urysohn Metrization Theorem would imply that X is metrizable.

($\implies$) Conversely, suppose that X is a metrizable with a metric d. Then for $x \in X$, consider the following collection of open balls centered at x

$$\mathscr{B}_x = \{B_d(x, 1/n) \,|\, n \in \mathbb{N}\}.$$

Note that $\mathscr{B}_x$ forms a countable local basis at x. Now consider the collection of open sets

$$\mathscr{U} = \bigcup_{x \in X} \mathscr{B}_x.$$

Clearly, $\mathscr{U}$ is an open cover for X. Since X is compact, there exists a finite subcover $\{B_d(x_1, 1/n_1), \ldots, B_d(x_k, 1/n_k)\}$ for X. It is not hard to see (why?) that the countable collection of open balls

$$\mathscr{B} = \bigcup_{i=1}^{n} \mathscr{B}_{x_i}$$

is a basis for the topology on X. Hence, X is second countable.